Quantum Error Correction of Time-Correlated Errors

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Abstract

The complexity of the error correction circuitry forces us to design quantum error correction codes capable of correcting a single error per error correction cycle. Yet, time-correlated error are common for physical implementations of quantum systems; an error corrected during the previous cycle may reoccur in the next cycle due to physical processes specific for each physical implementation of the qubits. In this paper we study quantum error correction for a restricted class of time-correlated errors in a spin-boson model. The algorithm we propose allows the correction of two errors per error correction cycle, provided that one of them time-correlated. The algorithm can be applied to any quantum error correcting code when the two logical qubits $\mid 0_L\rangle$ and $\mid 1_L\rangle$ are entangled states of a 2^n basis states in $\mathcal{H}_{2^n}$.

1 Quantum Error Correction

Since the early days of computing, reliability has been a major concern. Knowing that quantum states are subject to decoherence, the question whether a reliable quantum computer could be built was asked early on. A "pure state" $\mid \varphi\rangle = \alpha_0 \mid 0\rangle + \alpha_1 \mid 1\rangle$ may be transformed as a result of the interaction with the environment into a "mixed state" with density matrix:

$$\rho = \mid \alpha_0 \mid^2 \mid 0\rangle\langle 0 \mid + \mid \alpha_1 \mid^2 \mid 1\rangle\langle 1 \mid .$$

Other forms of decoherence, e.g. leakage may affect the state probability amplitude as well.

The initial thought was that a quantum computation could only be carried out successfully if its duration is shorter than the decoherence time of the quantum computer. As we shall see in Section 2, the decoherence time ranges from about 10^4 seconds for the nuclear spin embodiment of a qubit, to 10^{-9} seconds for quantum dots based upon charge. Thus, it seemed very problematic that a quantum computer could be built unless we have a mechanism to deal periodically with errors. Now we know [19] that quantum error correcting codes can be used to ensure fault-tolerant quantum computing; quantum error correction allows us to deal algorithmically with decoherence. There is a significant price to pay to achieve fault-tolerance through error correction: the number of qubits required to correct errors could be several orders of magnitude larger than the number of "useful" qubits [7].

In 1996, Shor [19] showed how to perform reliable quantum computations when the probability of a qubit or quantum gate error decays polylogarithmically with the size of the computation, a rather unrealistic assumption. The "quantum threshold theorem" ensures that that arbitrary long computations can be carried out with high reliability provided that error rate is below an *accuracy threshold* according to Knill, Laflamme, and Zurek [12]. In 1999, Aharonow and Ben-Or [1] proved that reliable computing is possible when the error rate is smaller than a constant threshold, but the cost is polylogarithmic in time and space. In practice, error correction is successful for a quantum system whose decoherence time is four to five orders of magnitude larger than the *gate time*, the time required for a quantum gate operation [7].

We describe the effect of the environment upon a qubit as a transformation given by Pauli operators: (i) the state of the qubit is unchanged if we apply the σ_I operator; (ii) a bit-flip error is the result of applying the transformation given by σ_x; (iii) a phase-flip error is the result of applying the transformation given by σ_z; and (iv) a bit-and-phase flip error is the result of applying the transformation given by $\sigma_y = i\sigma_x\sigma_z$.

A quantum error correcting scheme takes advantage of entanglement in two ways:

- We entangle one qubit carrying information with $(n-1)$ other qubits initially in state $\mid 0\rangle$ and create an n-qubit quantum codeword which is more resilient to errors.
- We entangle the n qubits of the quantum codeword with ancilla qubits in such a way that we can measure the ancilla qubits to determine the error syndrome without altering the state of the n-qubit codeword, by performing a so called non demolition measurement. The error syndrome tells if the individual qubits of the codeword have been affected by errors as well as the type of error.
- Finally, we correct the error(s).

Even though the no-cloning theorem prohibits the replication of a quantum state, we are able to encode a single logical qubit as multiple physical qubits and thus we can correct quantum errors [21]. For example, we can encode the state of a qubit

$$\mid \psi\rangle = \alpha_0 \mid 0\rangle + \alpha_1 \mid 1\rangle$$

as a linear combination of $\mid 00000\rangle$ and $\mid 11111\rangle$:

$$\mid \varphi\rangle = \alpha_0 \mid 00000\rangle + \alpha_1 \mid 11111\rangle.$$

Alternatively, we can encode the qubit in state $\mid \psi\rangle$ as:

$$\mid \varphi\rangle = \alpha_0 \mid 0_L\rangle + \alpha_1 \mid 1_L\rangle,$$

with $\mid 0_L\rangle$ and $\mid 1_L\rangle$ expressed as a superposition of codewords of a classical linear code. In this case all codewords are superpositions of vectors in $\mathcal{H}_{2^5}$, a Hilbert space of dimension 2^5. Steane's seven-qubit code [20] and Shor's nine qubit code [4] are based upon this scheme.

When we use the first encoding scheme, a random error can cause departures from the subspace spanned by $\mid 00000\rangle$ and $\mid 11111\rangle$. We should be able to correct small bit-flip errors because the component which was $\mid 00000\rangle$ is likely to remain in a sub-space $\mathcal{C}_0 \subset \mathcal{H}_{2^5}$ spanned by the six vectors:

$$\mid 00000\rangle, \mid 00001\rangle, \mid 00010\rangle, \mid 00100\rangle, \mid 01000\rangle, \mid 10000\rangle$$

while the component which was $\mid 11111\rangle$ is likely to remain in a sub-space $\mathcal{C}_1 \subset \mathcal{H}_{2^5}$ spanned by the six vectors,

$$\mid 11111\rangle, \mid 11110\rangle, \mid 11101\rangle, \mid 11011\rangle, \mid 10111\rangle, \mid 01111\rangle.$$

The two subspaces are disjoint:

$$\mathcal{C}_0 \cap \mathcal{C}_1 = \varnothing$$

thus we are able to correct a bit-flip error of any single physical qubit. This procedure is reminiscent of the basic idea of classical error correction when we determine the Hamming sphere an n-tuple belongs to, and then correct it as the codeword at the center of the sphere.

A quantum code encodes one logical qubit into n physical qubits in the Hilbert space $\mathcal{H}_{2^n}$ with basis vectors $\{\mid 0\rangle, \mid 1\rangle, \ldots \mid i\rangle, \ldots \mid 2^n - 1\rangle\}$. The two logical qubits are entangled states in $\mathcal{H}_{2^n}$:

$$| 0_L\rangle = \sum_i \alpha_i \mid i\rangle \qquad \text{and} \qquad | 1_L\rangle = \sum_i \beta_i \mid i\rangle$$

A quantum error correcting code must map coherently the two-dimensional Hilbert space spanned by $| 0_L\rangle$ and $| 1_L\rangle$ into two-dimensional Hilbert spaces corresponding to bit-flip, phase-flip, as well as bit-and-phase flip of each of the n qubits to ensure that the code is capable of correcting the three types of error for each qubit. Thus, the Hilbert space $\mathcal{H}_{2^n}$ must be large enough to accommodate n two-dimensional subspaces corresponding to bit-flip errors of each of the n qubits, n subspaces for the phase-flip errors, and n subspaces for bit-flip and phase-flip errors, and finally one subspace for the entangled state $| 0_L\rangle$ without errors. A similar requirement for $| 1_L\rangle$ leads us to conclude that n must satisfy the following inequality:

$$2(3n+1) \leq 2^n.$$

This inequality established by Laflamme, Miquel, Paz, and Zurek in [13] allows us to say that $n = 5$ is the smallest number of qubits required to encode the two superposition states $| 0_L\rangle$ and $| 1_L\rangle$, and then be able to recover them regardless of the qubit in error and the type of errors. Steane's seven-qubit and Shor's nine qubit codes satisfy this inequality.

The same paper [13] introduces a family of 5-qubit quantum error correcting codes. The code $\mathcal{Q}$ with the logical codewords

$$\begin{aligned} | 0_L\rangle = \tfrac{1}{4} \quad & (| 00000\rangle + | 10010\rangle + | 01001\rangle + | 10100\rangle + | 01010\rangle - | 11011\rangle - | 00110\rangle - | 11000\rangle \\ & - | 11101\rangle - | 00011\rangle - | 11110\rangle - | 01111\rangle - | 10001\rangle - | 01100\rangle - | 10111\rangle + | 00101\rangle) \end{aligned}$$

$$\begin{aligned} | 1_L\rangle = \tfrac{1}{4} \quad & (| 11111\rangle + | 01101\rangle + | 10110\rangle + | 01011\rangle + | 10101\rangle - | 00100\rangle - | 11001\rangle - | 00111\rangle \\ & - | 00010\rangle - | 11100\rangle - | 00001\rangle - | 10000\rangle - | 01110\rangle - | 10011\rangle - | 01000\rangle + | 11010\rangle) \end{aligned}$$

is a member of this family. The code $\mathcal{Q}$ is a *perfect quantum error correcting code* because a logical codeword consists of the smallest number n of qubits to satisfy the inequality from [13]. Recall that a *perfect linear code* $[n, k, d]$ with $d = 2e+1$ is one where Hamming spheres of radius e are all disjoint and exhaust the entire space of n-tuples.

Different physical implementations reveal that the interactions of the qubits with the environment are more complex and force us to consider spatially- as well as, time-correlated errors. If the qubits on an n qubit register are confined to a 3D structure, an error affecting one qubit will propagate to the qubits in a volume centered around the qubit in error. Spatially-correlated errors and means to deal with the spatial noise are analyzed in [3, 5, 11]. An error affecting qubit i of an n-qubit register may affect other qubits of the register. An error affecting qubit i at time t and corrected at time $t + \Delta$ may have further effect either on qubit i or on other qubits of the register.

Two major questions pertinent to error correction of correlated errors need to be addressed: first, can the quantum threshold theorem be extended for non-Markovian noise, and second, what are the practical means to deal with correlated errors. In this paper we only address the second question and restrict our discussion on how to deal with time-correlated errors.

The quantum threshold theorem is based upon the assumption that the quantum noise is Markovian and provides the foundation for fault-tolerant computations in environments where spatial and temporal correlations decay exponentially. Several recent papers [2, 16, 22] discuss fault-tolerant quantum computing in the presence of Non-Markovian noise. The technique used in [2, 22] to extend the threshold theorem for correlated environments is to establish an upper bound for the norm of the interaction Hamiltonian and for the error probability. Terhal and Burkhard [22] propose a model of decoherence where a qubit $| q_i\rangle$ is coupled to a bath B_i and when qubits $| q_i\rangle$ and $| q_j\rangle$ interact they are coupled to a common bath B_{ij}. They derive a threshold result for fault-tolerant quantum computation based upon the assumption that the interaction Hamiltonian between the qubits and the bath is a sum of terms and each term couples a single qubit with some portion of the bath. Aharonov, Kitaev, and Preskill [2] consider a time dependent Hamiltonian which describes the evolution of the system and the bath and make a distinction between "short-range" and "long-range" noise. A different approach to deal with correlated errors is considered in [16] where the theory of quantum phase transition is used to restate the threshold theorem as a dimensional criterion. The error model considered in this paper is based upon a recent study which addresses the problem of reliable quantum computing using solid state systems [15].

There is a conceptual difference between classical and quantum error correcting codes. We design a classical error correcting code to correct a number e of errors. The number of errors e is dictated by the physical properties of the communication channel, or of the storage device. It makes no difference if the errors are correlated, as long as the total number of errors is less or equal with the error correction capability of the code.

On the other hand, we determine the period of an error correction cycle for a quantum code to ensure that the probability of a qubit to be in error, ϵ, is very small, $10^{-5} \leq \epsilon \leq 10^{-4}$. Thus, the probability of e or more errors is very small indeed, $\mathcal{O}(\epsilon^e)$, and can be ignored. For this reason we are primarily interested in quantum error correcting codes capable to correct a single error.

Correlating could cause two or more errors to occur during an error correction cycle. There are two obvious approaches to deal with this problem:

- (i) design a code capable to correct these two or more errors, or
- (ii) use the classical information regarding past errors and quantum error correcting codes capable to correct a single error.

When a quantum error correcting code uses a large number of qubits and quantum gates it becomes increasingly more difficult to carry out the encoding and syndrome measurements during a single quantum error correction cycle. For example, if we encode a logical qubit using Shor's nine qubit code we need 9 physical qubits. If we use a two-level convolutional code based upon Shor's code then we need 81 physical qubits and $63 = 7 \times 9$ ancilla qubits to ensure that the circuit for syndrome calculation is fault-tolerant. While constructing quantum codes capable of correcting a larger number of errors is possible, we believe that the price to pay, the increase in circuit complexity makes this solution undesirable; this motivates our interest in the second approach.

2 Time-Correlated Quantum Errors

Figure 1 illustrates the evolution in time of two qubits, i and j for two error correction cycles. The first error correction cycle ends at time t_2 and the second at time t_5. At time t_1 qubit i is affected by decoherence and flips; at time t_2 it is flipped back to its original state during the first error correction step; at time t_3 qubit j is affected by decoherence and it is flipped; at time t_4 the correlation effect discussed in this section affects qubit i and flips it to an error state. If an algorithm is capable to correct one correlated error in addition to a "new" error, then during the second error correction the errors affecting qubits i and j are corrected at time t_5.

The quantum computer and the environment are entangled during the quantum computation. When we measure the state of the quantum computer this entanglement is translated into a probability ϵ that the measured state differs from the expected one. This probability of error determines the actual number of errors a quantum code is expected to correct.

If τ_{gate} is the time required for a single gate operation and τ_{dch} is the decoherence time of a qubit, then n_{gates}, the number of gates that can be traversed by a register before it is affected by decoherence is given by:

$$n_{gates} = \frac{\tau_{dch}}{\tau_{gate}}.$$

Quantum error correction is intimately related to the physical processes which cause decoherence. Table 1 presents sample values of the time required for a single gate operation τ_{gate}, the decoherence time of a qubit, τ_{dch}, and n_{gates}, the number of gates that can be traversed before a register of qubits is affected by decoherence, for several qubit implementations [10, 14, 17, 24]. We notice a fair range

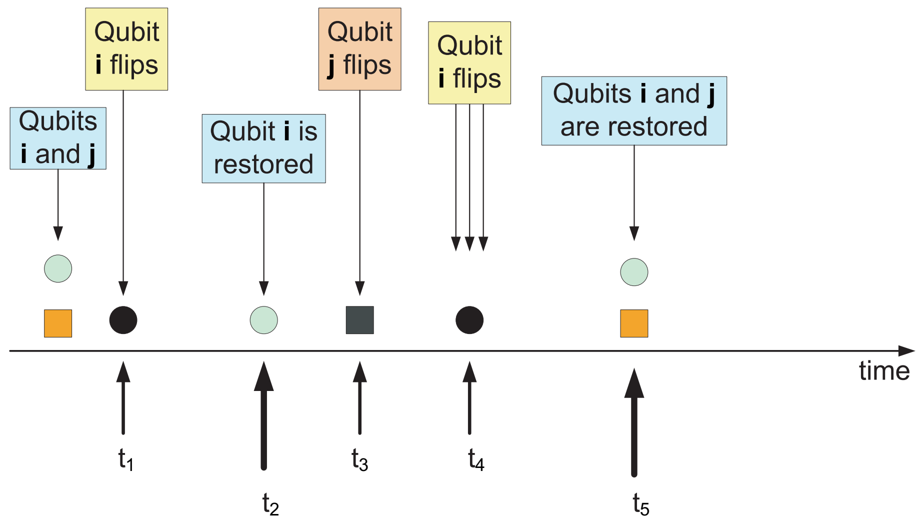

Figure 1: Time-correlated quantum errors. Two consecutive error correction cycles occurring at time t_2 and t_5 are shown. At time t_5 a code designed to correct a single error will fail while a code capable to handle time-correlated errors will correct the "new" error of qubit j and the "old" correlated error of qubit i.

Table 1: The time required for a single gate operation, τ_{gate}, the decoherence time of a qubit, τ_{dch}, and the number of gates that can be traversed before a register of qubits is affected by decoherence, n_{gates}.

Qubit implementation	$\tau_{dch(sec)}$	$\tau_{gate(sec)}$	n_{gates}
Nuclear spin	10^4	10^{-3}	10^7
Trapped Indium ion	10^{-1}	10^{-14}	10^{13}
Quantum dots/charge	10^{-9}	10^{-12}	10^3
Quantum dots/spin	10^{-6}	10^{-9}	10^3
Optical cavity	10^{-5}	10^{-14}	10^9

of values for the number of quantum gate operations that can be performed before decoherence affects the state.

The information in Table 1, in particular the decoherence time, can be used to determine the length of an error correction cycle, the time elapsed between two consecutive error correction steps. The number of quantum gate operations limits the complexity of the quantum circuit required for quantum error correction.

The Quantum Error Correction theory is based upon the assumption that the quantum system has a constant error rate ϵ. This implies that once we correct an error at time t_c, the system behavior at time $t > t_c$ is decoupled from events prior to t_c.

The Markovian error model is violated in physical implementations of qubits; for example, in a recent paper Novais and Baranger [15] discuss the decoherence in a spin-boson model which is applicable , for instance, to quantum dots. The authors assume that the qubits are perfect, thus eventual errors are due to dephasing and consider a linear coupling of the qubits to an ohmic bath. Using this model the authors analyze the three qubit Steane's code. They calculate the probability of having errors in quantum error correction cycles starting at times t_1 and t_2 and show that the probability of errors consists of two terms; the first is the uncorrelated probability and the second is the contribution due to correlation between errors in different cycles (Δ is the period of the error correcting cycle):

$$P \approx (\frac{\epsilon}{2})^2 + \frac{\lambda^4 \Delta^4}{8(t_1 - t_2)^4}$$

We see that correlations in the quantum system decay algebraically in time, and the latest error will influence the system most.

Only phase-flip errors are discussed in detail in [15]. The approach introduced by the authors is very general and can be extended to include other types of errors. Unfortunately, due to the noncommutative nature of phase and bit-flip events, the calculations become quite involved [23]. Nevertheless, one does expect major departures from the results presented in [15].

An important question for the error model discussed in [15] is the independence of phase- and bit-flip errors, namely, can a phase-flip error be correlated to a posterior bit-flip error? The answer will depend on the particular environment one tries to model. For solid-state qubits, these errors are caused by different bath modes which may not efficiently communicate with each other (such as for phonon-induced tunnelling and bias fluctuations in quantum dot charge qubits, which cause bit- and phase-flip errors, respectively). In this case, distinct errors are expected to be very weakly correlated in time and space while correlated errors are mostly of the same kind.

3 Stabilizer Codes

The stabilizer formalism is a succinct manner of describing a quantum error correcting code by a set of quantum operators [8]. We first review several concepts and properties of stabilizer codes.

The 1-qubit Pauli group $\mathcal{G}_1$ consists of the Pauli operators, σ_I, σ_x, σ_y, and σ_z together with the multiplicative factors ± 1 and $\pm i$:

$$\mathcal{G}_1 \equiv \{\pm\sigma_I, \pm i\sigma_I, \pm\sigma_x, \pm i\sigma_x, \pm\sigma_y, \pm i\sigma_y, \pm\sigma_z, \pm i\sigma_z\}.$$

The generators of $\mathcal{G}_1$ are:

$$\langle \sigma_x, \sigma_z, i\sigma_I \rangle.$$

Indeed, every element of $\mathcal{G}_1$ can be expressed as a product of a finite number of generators. For example:

$$-\sigma_x = i\sigma_I i\sigma_I \sigma_x, \qquad +i\sigma_x = i\sigma_I \sigma_x, \qquad -i\sigma_x = i\sigma_I i\sigma_I i\sigma_I \sigma_x.$$

The n-qubit Pauli group $\mathcal{G}_n$ consists of the 4^n tensor products of σ_I, σ_x, σ_y, and σ_z with an overall phase of ± 1 or $\pm i$. Elements of the group can be used to describe the error operators applied to an n-qubit register. The *weight* of such an operator in $\mathcal{G}_n$ is equal to the number of tensor factors which are not equal to σ_I.

For example, consider the case $n = 5$. The 5-qubit Pauli group consists of the tensor products of the form:

$$\mathbf{E}^{(1)} \otimes \mathbf{E}^{(2)} \otimes \mathbf{E}^{(3)} \otimes \mathbf{E}^{(4)} \otimes \mathbf{E}^{(5)} \quad \text{with} \quad \mathbf{E}^{(i)} \in \mathcal{G}_1, \quad 1 \leq i \leq 5.$$

The operator:

$$\mathbf{E}_\alpha = \sigma_I \otimes \sigma_x \otimes \sigma_I \otimes \sigma_z \otimes \sigma_I$$

has a weight equal to 2 and represents a bit-flip of qubit 2 and a phase-flip of qubit 4 where the qubits are numbered from left to right.

The stabilizer S of code $\mathcal{Q}$ is a subgroup of the n-qubit Pauli group,

$$S \subset \mathcal{G}_n.$$

The generators of the subgroup S are:

$$M = \{\mathbf{M}_1, \mathbf{M}_2 \dots \mathbf{M}_q\}.$$

The eigenvectors of the generators $\{\mathbf{M}_1, \mathbf{M}_2 \dots \mathbf{M}_q\}$ have special properties: those corresponding to eigenvalues of $+1$ are the codewords of $\mathcal{Q}$ and those corresponding to eigenvalues of -1 are codewords affected by errors. If a vector $\mid \psi_i\rangle \in \mathcal{H}_n$ satisfies,

$$\mathbf{M}_j \mid \psi_i\rangle = (+1) \mid \psi_i\rangle \qquad \forall \mathbf{M_j} \in M$$

then $\mid \psi_i\rangle$ is a codeword, $\mid \psi_i\rangle \in \mathcal{Q}$. This justifies the name given to the set S, any operator in S stabilizes a codeword, leaving the state of a codeword unchanged. On the other hand if:

$$\mathbf{M}_j \mid \varphi_k\rangle = (-1) \mid \varphi_k\rangle.$$

then $\mid \varphi_k\rangle = \mathbf{E}_i \mid \psi_k\rangle$, the state $\mid \varphi_k\rangle$ is a codeword $\mid \psi_k\rangle \in \mathcal{Q}$ affected by error $\mathbf{E}_i$. The error operators affecting codewords in $\mathcal{Q}$, $E = \{\mathbf{E}_1, \mathbf{E}_2 \dots\}$, are also a subgroup of the n-qubit Pauli group

$$E \subset \mathcal{G}_n.$$

Each error operator $\mathbf{E}_i$ is a tensor product of n Pauli matrices. Its weight is equal to the number of errors affecting a quantum work, thus the number of Pauli matrices other than σ_I.

The coding space:

$$Q = \{\mid \psi_i\rangle \in \mathcal{H}_n \;\text{ such that }\; M_j \mid \psi_i\rangle = (+1) \mid \psi_i\rangle, \qquad \forall M_j \in S\ \}$$

is the space of all vectors $\mid \psi_i\rangle$ fixed by S

It is easy to prove that S is a stabilizer of a non-trivial Hilbert subspace $V_{2^n} \subset \mathcal{H}_{2^n}$ if and only if:

1. $S = \{\mathbf{S}_1, \mathbf{S}_2, \dots\}$ is an Abelian group:

$$\mathbf{S}_i\mathbf{S}_j = \mathbf{S}_j\mathbf{S}_i, \;\; \forall \mathbf{S}_i, \mathbf{S}_j \in S \;\; i \neq j.$$

2. The identity matrix multiplied by -1 is not in S:

$$-(\sigma_I^{\otimes n}) \notin S.$$

If $\mathbf{E}$ is an error operator, and $\mathbf{E}$ anti-commutes with some element $\mathbf{M} \in S$, then $\mathbf{E}$ can be detected, since for any $\mid \psi_i\rangle \in Q$:

$$\mathbf{ME} \mid \psi_i\rangle = -\mathbf{EM} \mid \psi_i\rangle = -\mathbf{E} \mid \psi_i\rangle$$

Definition. *The normalizer $N(S)$ consists of operators $\mathbf{E} \in \mathcal{G}_n$ such that $\mathbf{E}\mathbf{S}_i\mathbf{E}^\dagger \in S, \forall \mathbf{S}_i \in S$. The distance d of a stabilizer code is the minimum weight of an element in $N(S) - S$ with $N(S)$ the normalizer of S. An $[n, k, d]$ stabilizer code is an $[n, k]$ code stabilized by S and with distance d; n is the length of a codeword, and k the number of information symbols.*

An $[n, k, d]$ stabilizer code with the minimum distance $d = 2e + 1$ can correct at most e arbitrary quantum errors or $2e$ errors whose location is well known. In [6] this property of a stabilizer code is used for syndrome decoding for optical cluster state quantum computation.

Given an $[n, k, d]$ stabilizer code the cardinalities of the stabilizer S and of its generator M are:

$$\mid S \mid = 2^{n-k}, \qquad \mid M \mid = n - k$$

The error syndrome corresponding to the stabilizer $\mathbf{M}_j$ is a function of the error operator, $\mathbf{E}$, defined as:

$$f_{\mathbf{M}_j}(\mathbf{E}) : \mathcal{G} \mapsto \mathbf{Z}_2 \qquad f_{\mathbf{M}_j}(\mathbf{E}) = \begin{cases} 0 & \text{if } [\mathbf{M}_j, \mathbf{E}] = 0 \\ 1 & \text{if } \{\mathbf{M}_j, \mathbf{E}\} = 0, \end{cases}$$

where $[\mathbf{M}_j, \mathbf{E}]$ is the commutator and $\{\mathbf{M}_j, \mathbf{E}\}$ the anti-commutator of operators $\mathbf{M}_j$ and $\mathbf{E}$. Let $f(\mathbf{E})$ be the $(n-k)$-bit integer given by the binary vector:

$$f(\mathbf{E}) = (f_{\mathbf{M}_1}(\mathbf{E}) f_{\mathbf{M}_2}(\mathbf{E}) \ldots f_{\mathbf{M}_{n-k}}(\mathbf{E})).$$

This $(n-k)$-bit integer is called the *syndrome of error* $\mathbf{E}$.

Proposition. *The error syndrome uniquely identifies the qubit(s) in error if and only if the subsets of the stabilizer group which anti-commute with the error operators are distinct.*

An error can be identified and corrected only if it can be distinguished from any other error in the error set. Let $Q(S)$ be the stabilizer code with stabilizer S. The *Correctable Set of Errors* for Q(S) includes all errors which can be detected by S and have distinct error syndromes.

Corolary. *Given a quantum error correcting code Q capable to correct e_u errors, the syndrome does not allow us to distinguish the case when more than e_u qubits are in error. When we have exact prior knowledge about e_c correlated errors the code is capable of correcting these $e_u + e_c$ errors.*

Proof: Assume that F_1, F_2 cause at most e_u qubits to be in error, thus F_1, F_2 are included in the *Correctable Set of Errors* of Q. Assuming errors F_1 and F_2 are distinguishable, there must exist some operator $M \in S$ which commutes with one of them, and anti-commutes with the other:

$$F_1^T F_2 M = -M F_1^T F_2$$

If we know the exact correlated errors E in the system, then:
$(E^T F_1)^T (E^T F_2) M = (F_1^T E E^T F_2) M = F_1^T F_2 M = -M F_1^T F_2 = -M (E^T F_1)^T (E^T F_2)$
which means that the stabilizer M commutes with one of the two errors $E^T F_1$, $E^T F_2$ and anti-commutes with the other. So error $E^T F_1$ is distinguishable from error $E^T F_2$. Therefore, if we know the exact prior errors E, we can identify and correct any $E^T F_i$ errors with the weight of F_i equal or less than e_u.

For example, consider a 5-qubit quantum error-correcting code $\mathcal{Q}$ with $n = 5$ and $k = 1$ from [13] discussed in Section 1. The stabilizer S of this code is described by a group of 4 generators:

$$M = \{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_2\} \quad \begin{array}{l} \text{with the generators:} \\ \mathbf{M}_1 = \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x \otimes \sigma_I, \quad \mathbf{M}_2 = \sigma_I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x, \\ \mathbf{M}_3 = \sigma_x \otimes \sigma_I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z, \quad \mathbf{M}_4 = \sigma_z \otimes \sigma_x \otimes \sigma_I \otimes \sigma_x \otimes \sigma_z. \end{array}$$

It is easy to see that two codewords are eigenvectors of the stabilizers with an eigenvalue of (+1):

$$\mathbf{M}_j \mid 0_L\rangle = (+1) \mid 0_L\rangle \quad \text{and} \quad \mathbf{M}_j \mid 1_L\rangle = (+1) \mid 1_L\rangle, \quad 1 \leq j \leq 4.$$

Note also that M is an Abelian subgroup, each generator commutes with all the others.

Table 2 lists single error operators and the generator(s) which anti-commute with each operator. For example, $\mathbf{X}_1$ anti-commutes with $\mathbf{M}_4$, thus a bit-flip on the first qubit can be detected; $\mathbf{Z}_1$ anti-commutes with $\mathbf{M}_1$ and $\mathbf{M}_3$, thus a phase-flip of the first qubit can also be detected. Since each of these 15 errors anti-commute with distinct subsets of S we can distinguish individual errors and then correct them. An example shows that the code cannot detect two qubit errors; indeed, the two bit-flip error

Table 2: Single error operators for the 5-qubit code and the generator(s) that anti-commute with each operator. Subscripts indicate positions of errors, e.g. X_3 means a bit-flip error on the third qubit, $X_3 = \sigma_I \otimes \sigma_I \otimes \sigma_x \otimes \sigma_I \otimes \sigma_I$.

Error operator	Generator(s)	Error operator	Generator(s)	Error operator	Generator(s)
$\mathbf{X}_1$	$\mathbf{M}_4$	$\mathbf{Z}_1$	$\mathbf{M}_1,\ \mathbf{M}_3$	$\mathbf{Y}_1$	$\mathbf{M}_1,\ \mathbf{M}_3,\ \mathbf{M}_4$
$\mathbf{X}_2$	$\mathbf{M}_1$	$\mathbf{Z}_2$	$\mathbf{M}_2,\ \mathbf{M}_4$	$\mathbf{Y}_2$	$\mathbf{M}_1,\ \mathbf{M}_2,\ \mathbf{M}_4$
$\mathbf{X}_3$	$\mathbf{M}_1,\ \mathbf{M}_2$	$\mathbf{Z}_3$	$\mathbf{M}_3$	$\mathbf{Y}_3$	$\mathbf{M}_1,\ \mathbf{M}_2,\ \mathbf{M}_3$
$\mathbf{X}_4$	$\mathbf{M}_2,\ \mathbf{M}_3$	$\mathbf{Z}_4$	$\mathbf{M}_1,\ \mathbf{M}_4$	$\mathbf{Y}_4$	$\mathbf{M}_1,\ \mathbf{M}_2,\ \mathbf{M}_3,\ \mathbf{M}_4$
$\mathbf{X}_5$	$\mathbf{M}_3,\ \mathbf{M}_4$	$\mathbf{Z}_5$	$\mathbf{M}_2$	$\mathbf{Y}_5$	$\mathbf{M}_2,\ \mathbf{M}_3,\ \mathbf{M}_4$

$$\mathbf{X}_1\mathbf{X}_2 = \sigma_x \otimes \sigma_x \otimes \sigma_I \otimes \sigma_I \otimes \sigma_I$$

is indistinguishable from $\mathbf{Z}_4 = \sigma_I \otimes \sigma_I \otimes \sigma_I \otimes \sigma_z \otimes \sigma_I$ because both $\mathbf{X}_1\mathbf{X}_2$ and $\mathbf{Z}_4$ anti-commute with the same subset of stabilizers, $\{\mathbf{M}_1, \mathbf{M}_4\}$, and give the same error syndrome. Therefore, *the 5-qubit code can correct any single qubit error, but cannot correct two qubit errors.*

If we know the correlated error in the system, for example, $\mathbf{X}_3$, then, from Table 2, it is easy to see that the errors $\mathbf{X}_3\mathbf{E}_i$ have distinct error syndromes for $1 \leq i \leq 15$. Therefore we can identify these errors and correct them.

4 Using Stabilizer Codes to Correct Time-Correlated Errors

Classical, as well as quantum error correction schemes allow us to construct codes with a well-defined error correction capability. If a code is designed to correct e errors, it will fail whenever more than e errors occur.

From previous section we know that if time-correlated errors are present in the system, we can not always detect them through the calculation of the syndrome. The same syndrome could signal the presence of a single error or more errors, as shown in the above example. But, if we maintain a history of previous errors in the system, we can remove this ambiguity.

In this section we extend the error correction capabilities of any code designed to correct a single error (bit-flip, phase-flip error, or bit-and-phase flip) and allow the code to correct an additional time-correlated error. In addition to standard assumptions for quantum error correction,

- Quantum gates are perfect and operate much faster than the characteristic response of the environment;
- The states of the computer can be prepared with no errors;

we also assume that:

- There is no spatial error correlation, which means a qubit in error does not influence its neighbors;
- In each error correcting cycle, there is one new error E_a which occurs with a constant probability ε_a, and there is also a time-correlated error E_b with probability $\varepsilon_b(t)$. As correlations decay in time, the error which occurred last and has been corrected in the previous error correction cycle, has a high probability to re-occur; the time-correlated error E_b affects the same qubit and is of the same type (bit-flip, phase-flip, or bit-and-phase flip) as the new error corrected during the previous error correction cycle;

- A classical circuit stores information about the last qubit in error and controls a quantum circuit to entangle one additional ancilla qubit with the qubit affected by error during the last error correction cycle.

Quantum error correction requires non-demolition measurements of the error syndrome in order to preserve the state of the physical qubits. In other words, a measurement of the probe (the ancilla qubits) should not influence the free motion of the signal system. The syndrome has to identify precisely the qubit(s) in error and the type of error(s). Thus, the qubits of the syndrome are either in the state with density $\mid 0\rangle\langle 0 \mid$ or in the state with density $\mid 1\rangle\langle 1 \mid$, which represent classical information.

A quantum non-demolition measurement allows us to construct the error syndrome $\Sigma_{current}$. After examining the syndrome $\Sigma_{current}$ an error correcting algorithm should be able to decide whether:

1. No error has occurred, and in that case no action should be taken;
2. One new error, E_a, has occurred. We apply the corresponding Pauli transformation to the qubit in error;
3. Two or more errors have occurred. In this case there are two distinct possibilities: (a) We have a new error and the last error we have corrected in the previous cycle has re-occurred; (b) There are two or more new errors. A quantum error correcting code capable of correcting a single error in one error correction cycle will fail in both cases. We now describe an algorithm which is able to handle case (a).

It is rather hard to distinguish the last two possibilities. For perfect codes, the syndrome S_{ab} corresponding to two errors, E_a and E_b, is always identical to the syndrome S_c for some single error E_c. The stabilizer formalism does not allow us to distinguish a double error from a single one. For a non-perfect code it is sometimes possible to distinguish the two syndromes.

The syndrome manipulation of any quantum error correction algorithm should not alter the state of the original qubits. For this reason we introduce an additional ancilla qubit and entangle it with the qubit in error in the last error correction cycle. Thus the additional ancilla qubit becomes entangled with the n qubits of the codeword. Our scheme then allows us to compute the syndrome S_{n+1} corresponding to an extended codeword as well as the original syndrome S_n. If the two syndromes are different, it indicates the existence of the time correlated error. Then we can identify and correct the new error together with the time correlated one.

We consider a stabilizer code with a codeword consisting of n qubits and k ancilla qubits for the syndrome calculation. We assume that a classical circuit holds the syndrome Σ_{last} of a time-correlated error, an error corrected during the previous error correction cycle, which may re-occur during the current cycle.

The error correction algorithm requires two groups of k-ancilla qubits and one additional ancilla qubit; the ancilla qubits are initially in state $\mid 0\rangle$. Our algorithm handles the three possible error types affecting a qubit during the previous error correcting cycle in different manners: (1) bit-flip; (2) bit-and-phase flip; and (3) phase-flip. We now discuss each of these three cases.

1. Correction of bit-flip time-correlated error. The quantum circuit for correction of time-correlated errors uses a `CNOT` gate to entangle the qubit in error with the additional ancilla qubit, then compute and measure a syndrome, S_{n+1} for an $(n+1)$ extended codeword consisting of the original codeword and the additional ancilla qubit; the first group of k ancilla qubits will be used for syndrome S_{n+1}. A second `CNOT` gate will disentangle the additional qubit from the n qubit codeword and a second syndrome S_n for the original codeword will be computed by using the second group of k ancilla qubits. The algorithm to correct a bit-flip time-correlated error consists of the following steps:

1. Entangle the qubit in error in the last error-correction cycle with the additional ancilla qubit by using a `CNOT gate`. In this process the additional ancilla qubit becomes entangled with the n qubits of the codeword.

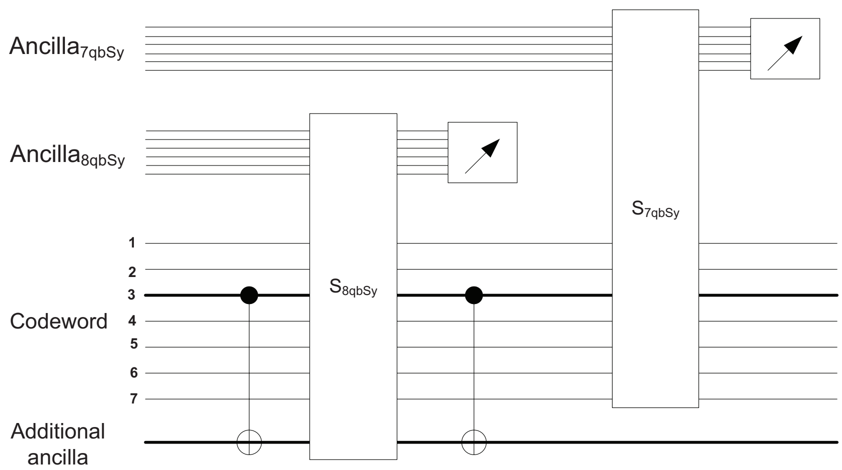

Figure 2: A quantum circuit for correction of a bit-flip time-correlated error for the Steane code. Qubit 3 was affected by a bit-flip error during the previous error correction cycle, and may re-occur. This qubit is entangled with the additional ancilla qubit, initially in state $| 0\rangle$, using a `CNOT` gate; the additional ancilla qubit becomes entangled with all seven qubits of the codeword. The ancilla qubits $Ancilla_{8qbSy}$ are used for the extended syndrome measurement S_{8qbSy} of the eight-qubit codeword. A second `CNOT` gate disentangles the additional ancilla qubit from the original Steane code. $Ancilla_{7qbSy}$ are the ancilla qubits used for the syndrome measurement S_{7qbSy} of the seven-qubit codeword after the additional ancilla qubit was disentangled from the seven qubits.

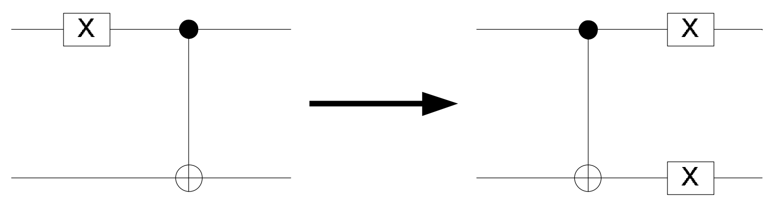

Figure 3: A CNOT gate duplicate the control qubit in Z basis and propagates a bit-flip of the control qubit to the target qubit

2. Measure the extended syndrome S_{n+1} for an $(n + 1)$ extended codeword consisting of the original codeword and an additional ancilla qubit. Call Σ_1 the result of a measurement of the syndrome S_{n+1}. If $\Sigma_1 \neq 00\ldots00$ then Σ_1 is the syndrome corresponding to the new error:

$$\Sigma_{new} = \Sigma_1.$$

   Indeed, if a bit-flip error of qubit i re-occurred, the `CNOT` gate propagates this error to the additional ancilla qubit(Figure 3), and the effects of these two bit-flips cancel each other in the extended syndrome measurement S_{n+1} as shown in Figure 4.

3. Disentangle the additional ancilla qubit from the n qubits of the original codeword. It is an operation performed by a second `CNOT` gate.

4. Measure the syndrome S_n of the original n qubit codeword. Call Σ_2 the result of a measurement of the syndrome S_n.

5. Compare Σ_1 and Σ_2:

   - If $\Sigma_1 = \Sigma_2$, correct the new error indicated by Σ_1.

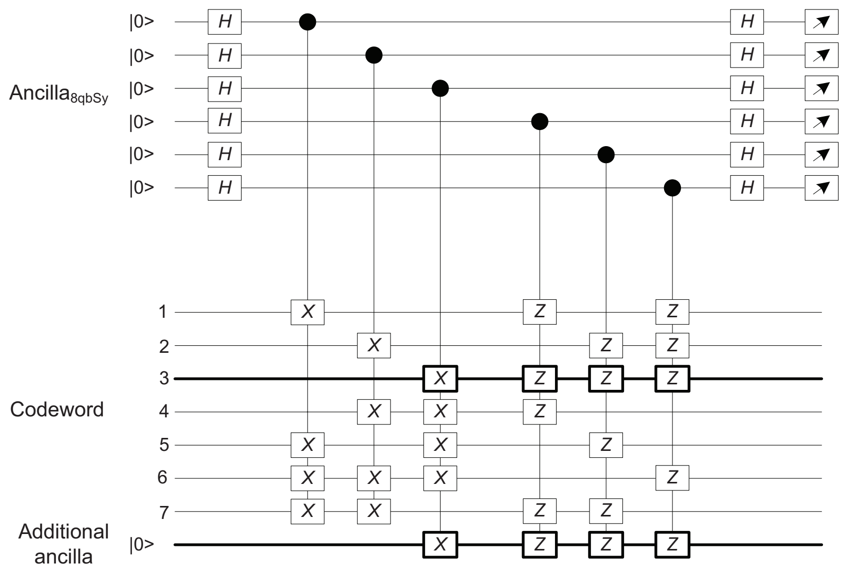

Figure 4: The quantum circuit to measure extended syndrome S_{8qbSy} in Figure 2. S_{8qbSy} is the syndrome of an eight-qubit codeword consisting of the seven qubits Steane code and the additional ancilla qubit. The top six ancilla qubits are used to measure the syndrome; the middle seven represent the codeword; the bottom qubit is the additional ancilla entangled with the qubit in error and, thus, with the entire group of seven qubits of the codeword. Bit-flips on qubit 3 and the additional ancilla qubit cancel each other's effect in the syndrome measurement.

- If $\Sigma_1 \neq \Sigma_2$, a time-correlated error may have occurred(check the syndrome table). Correct first the new error indicated by Σ_{new}; then use Σ_{last} to correct the time-correlated error.

Figures 2 and 4 show the circuit for an example based upon the seven-qubit Steane code with the last error is a bit-flip on qubit 3.

Example. *Consider the* 5*-qubit code discussed in Sections 1 and 3.*

The four ancilla qubits $a_1a_2a_3a_4$ of the syndrome Σ are computed using four generators, $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$, respectively. Table 3 lists the syndromes for each of the three types of error on one of the five qubits of a codeword. The 5-qubit code is a perfect code, and each of the 15 syndromes is associated with some single-qubit error. The syndrome 0000 corresponds to no error.

Table 3: The syndromes for each of the three types of errors on each qubit of a codeword for the 5-qubit code: X_{1-5} bit-flip, Z_{1-5} phase-flip, and Y_{1-5} bit-and-phase flip.

Error	Syndrome $a_1a_2a_3a_4$	Error	Syndrome $a_1a_2a_3a_4$	Error	Syndrome $a_1a_2a_3a_4$
X_1	0001	Z_1	1010	Y_1	1011
X_2	1000	Z_2	0101	Y_2	1101
X_3	1100	Z_3	0010	Y_3	1110
X_4	0110	Z_4	1001	Y_4	1111
X_5	0011	Z_5	0100	Y_5	0111

Assume that the error corrected during the last error correction cycle is a bit-flip of qubit 3. Then the syndrome $\mathbf{X}_3 = 1100$, and the corresponding information was recorded using classic bits. We use a quantum circuit similar to the one in Figure 2 for error correction. Two groups of four ancilla qubits are used to compute and measure the syndromes S_{6quSy} and S_{5quSy} as each codeword has only 5 qubits. If the error syndromes we measure are different, for example, $\Sigma_1 = 1101$ and $\Sigma_2 = 0001$, we conclude that the system has been affected by two or more errors. From Table 3 we notice that:

1. $\Sigma_1 = \mathbf{Y}_2$: The syndrome corresponds to a bit-flip and phase-flip error of qubit 2;
2. $\Sigma_2 = \mathbf{X}_3 \oplus \mathbf{Y}_2$: It confirms that the system has been subject to the time-correlated error, the bit-flip of qubit 3 with syndrome $\mathbf{X}_3$, in addition to a bit-and-phase flip error of qubit 2.

We correct the bit-and-phase flip error of qubit 2 and then the bit-flip of qubit 3.

2. Correction of bit-and-phase flip time-correlated error. In the last error correction cycle,a bit-flip and a phase-flip occurred on one qubit, named a bit-and-phase flip error. In this case if the time-correlated error occurred, the `CNOT` gate will propagate the bit-flip transformation to the additional ancilla, but not the phase-flip as shown in Figure 3. Thus the measurements of the two syndromes will always lead to $\Sigma_1 \neq \Sigma_2$. Assume Σ_{last} is the error syndrome which indicates that qubit i is affected by the time-correlated error. Then if $\Sigma_1 \neq \Sigma_2$:

1. Knowing Σ_{last}, the syndrome of the error corrected during the last error correction cycle (bit-and-phase flip of qubit i), compute the syndrome of the new error as:
$$\Sigma_{new} = \Sigma_2 \oplus \Sigma_{last}.$$
2. Determine Σ'_{last} the syndrome corresponding to a phase-flip only on qubit i affected by the time-correlated error since the phase-flip has not be propagated to the additional qubit. Compute:
$$\Sigma'_1 = \Sigma_{new} \oplus \Sigma'_{last}$$
3. If the system has exactly the new error plus the time correlated error, we should have $\Sigma'_1 = \Sigma_1$. Correct the new error indicated by the syndrome Σ_{new} first and then the time correlated error Σ_{last}. Else, error correction is not feasible.

Example. *Consider again the 5-qubit perfect quantum code.* Assume that the time-correlated error is a bit-flip and a phase-flip of qubit 5. The corresponding syndrome is $\Sigma_{last} = \mathbf{Y}_5 = 0111$.

A circuit similar to the one in Figure 2, allows us to measure the syndromes $\Sigma_1 = 1110$ and $\Sigma_2 = 1101$. The time-correlated error $\mathbf{Y}_5$ could have occurred because $\Sigma_1 \neq \Sigma_2$. With the aid of Table 3 we compute:

- $\Sigma_{new} = \Sigma_2 \oplus \Sigma_{last} = 1101 \oplus 0111 = 1010$. Thus $\Sigma_{new} = 1010$, a phase-flip of qubit 1.
- Find out the syndrome corresponding to a phase flip of qubit 5, $\Sigma'_{last} = \mathbf{Z}_5 = 0100$. Compute $\Sigma'_1 = \Sigma_{new} \oplus \Sigma'_{last} = 1010 \oplus 0100 = 1110$.
- $\Sigma'_1 = \Sigma_1$. Correct the new error indicated by Σ_{new}, a phase-flip of qubit 1, and then the time correlated error, corresponding to Σ_{last}, the bit-and-phase flip of qubit 5.

3. Correction of phase-flip time-correlated error. A `CNOT` gate duplicates the value of the qubit in the Z basis, it does not propagate a phase-flip. To compute the extended syndrome S_{n+1} this time we change the basis from $\mid 0\rangle, \mid 1\rangle$ to $\mid +\rangle, \mid -\rangle$; we construct a circuit called `HCNOT`, consisting of a `CNOT` with the control qubit sandwiched between two `Hadamard` gates. This `HCNOT` gate duplicates the state of the control qubit in the X basis and propagates a phase-flip of the control qubit as a bit-flip of the target qubit as shown in Figure 5.

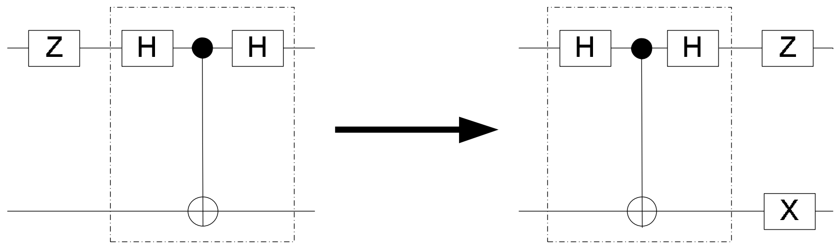

Figure 5: A HCNOT gate duplicates the control qubit in $|+\rangle, |-\rangle$ basis. It propagates a phase-flip of the control qubit as a bit-flip of the target qubit.

Example. *Consider again the Steane sever-qubit code.* Assume the time-correlated error is a phase-flip of qubit 3. We use a `HCNOT` gate to duplicate qubit 3 in the X basis and then a second `HCNOT` gate for disentanglement. Thus the additional ancilla qubit replicates the state of qubit 3 in the $|+\rangle, |-\rangle$ basis. To carry out the measurement of the extended syndrome, we have to replicate each action of qubit 3 to the additional ancilla qubit, but on the $|+\rangle, |-\rangle$ basis, to keep the entanglement between the additional qubit and the code. The new circuit for the extended syndrome measurement in Figure 6 is similar to the one the for the bit-flip time-correlated error in Figure 4. In this example a phase-flip of qubit 3 will propagate as a bit-flip on the additional ancilla qubit. Thus the effects of the phase-flip on qubit 3 and the bit-flip on the additional ancilla qubit will cancel each other in the extended syndrome measurement.

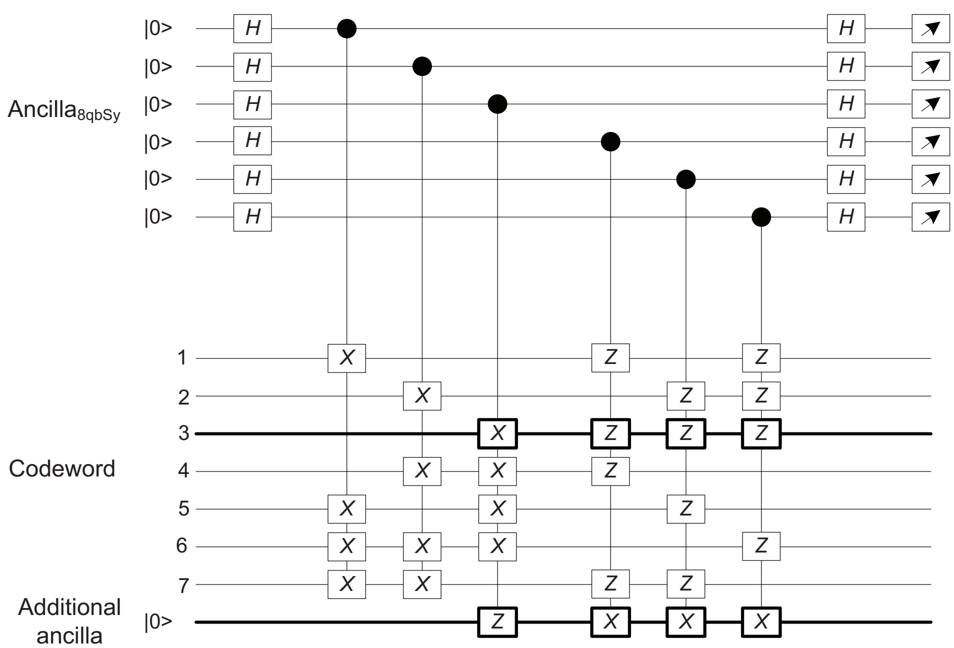

Figure 6: A quantum circuit for the measurement of the extended syndrome for the Stean seven-qubit quantum error correction code. The time correlated error is a phase-flip of qubit 3.

Example. *Consider the 5-qubit code.* Assume the time-correlated error is a phase-flip on qubit 1 with $\Sigma_{last} = \mathbf{Z}_1 = 1010$. A circuit similar to the one in Figure 6 allows us to measure $\Sigma_1 = 0100$. A following normal syndrome measurement gives $\Sigma_2 = 1110$. Since $\Sigma_1 \neq \Sigma_2$ and $\Sigma_2 \neq 0000$, we should expect the time-correlated error, as well as a new error.

It follows that $\Sigma_{new} = \Sigma_1 = \mathbf{Z}_5 = 0100$; the new error is a phase flip of qubit 5. We also see that $\Sigma_2' = \Sigma_{new} \oplus \Sigma_{last} = 0100 \oplus 1010 = 1110$. Our error correction scheme is consistent because $\Sigma_2' = \Sigma_2$. Thus, we correct the new error, a phase flip of qubit 5 with the syndrome $\Sigma_{new} = 0100$, then the time correlated error, a phase-flip on qubit 1 with $\Sigma_{last} = \mathbf{Z}_1 = 1010$.

The following algorithm summarizes the three cases discussed in this section:

1. Entangle qubit i, the qubit in error at the last error correction cycle identified by Σ_{last} with an additional ancilla qubit. Use a `CNOT` gate for a bit-flip or bit-and-phase-flip error indicated by Σ_{last}, or an `HCNOT` gate for a phase-flip.
2. Carry out a measurement of an extended syndrome S_{n+1} for an $(n+1)$-qubit; call Σ_1 the results of this measurement.
3. Disentangle the extra qubit from the codeword.
4. Carry out a syndrome measurement for the normal syndrome S_n of an n-qubit codeword; call Σ_2 the results of this measurement.
5. Compare Σ_1 and Σ_2:
   (a) If $\Sigma_1 = \Sigma_2 = 00\ldots0$, no error.
   (b) If $\Sigma_1 = \Sigma_2 \neq 00\ldots0$, the system is affected by a new error only, no time-correlated error. Correct the error according to the syndrome table.
   (c) If $\Sigma_1 \neq \Sigma_2$:
      i. If Σ_{last} indicates a bit-flip or a phase-flip error then $\Sigma_{new} = \Sigma_1$. Correct the new error with the syndrome Σ_{new} and then correct the time-correlated error with syndrome Σ_{last}.
      ii. If Σ_{last} indicates a bit-and-phase-flip error of qubit i then:
         - Compute the syndrome of the new error as: $\Sigma_{new} = \Sigma_2 \oplus \Sigma_{last}$.
         - Determine Σ_{last}' the syndrome corresponding to a phase-flip only of qubit i. Compute: $\Sigma_1' = \Sigma_{new} \oplus \Sigma_{last}'$.
         - If $\Sigma_1' = \Sigma_1$ correct first the new error indicated by the syndrome Σ_{new} and then the time correlated error Σ_{last}.
   (d) Otherwise, the system was subject to two or more errors that cannot be corrected.

5 Summary and Future Work

Errors in a realistic quantum system are not independent, often they are time-correlated. Based on the properties of time-correlated noise, we present an algorithm which allows the correction of one time-correlated error in addition to a new error. This algorithm needs two measurements of the syndrome and judges the system error depending on the syndromes.

The algorithms can be applied to any quantum error correcting code when the two logical qubits $| 0_L\rangle$ and $| 1_L\rangle$ are entangled states of a 2^n basis states in $\mathcal{H}_{2^n}$. The algorithms can be used for perfect, as well as for non-perfect quantum error correcting codes.

The algorithm requires quantum as well as classical circuits. It adds more complexity to the quantum error correction circuit because we need separate ancilla qubits for each evaluation of the syndrome. But, as we have mentioned earlier, the alternative is to use a quantum error correcting code capable of correcting two errors in one error correction cycles. This alternative will most likely require more complex encoding, decoding, and error correction quantum circuits than our approach.

Very recently, Lov Grover and Ben Reichardt [18] have proposed a radically new error correction scheme based upon Grover's fixed-point quantum search [9]. This approach could greatly simplify the correction of spatial and time-correlated quantum errors and we are investigating algorithms based upon these ideas.

6 Acknowledgments

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